

A BERRY-ESSEEN BOUND FOR L-STATISTICS
WITH UNBOUNDED WEIGHT FUNCTIONS

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A Berry-Esseen bound of order $n^{-\frac{1}{2}}$ is established for linear combinations of order statistics. The theorem extends previous results for the case of bounded weights to a class of L-statistics with unbounded weight functions.

1. INTRODUCTION AND RESULT

Let X_1, X_2, \dots, X_n be independent random variables (r.v.) with common distribution function (df) F and let $X_{1:n} \leq \dots \leq X_{n:n}$ be the corresponding order statistics. Let J be a fixed real-valued weight function on $(0,1)$. We consider L-statistics (or linear combinations of order statistics) of the form:

$$(1.1) \quad T_n = \sum_{i=1}^n \int_{\frac{i-1}{n}}^{i/n} J(s) ds X_{i:n}.$$

Let

$$(1.2) \quad F_n^*(x) = P(T_n^* \leq x) \quad \text{for } -\infty < x < \infty$$

where

$$(1.3) \quad T_n^* = (T_n - E(T_n)) / \sigma(T_n).$$

In the past decade there has been considerable interest into the asymptotic distribution theory for L-statistics. It is well-known that T_n^* is asymptotically normally distributed under quite general conditions. A survey of such results was given by Serfling (1980). We also refer to a recent paper of Mason (1981), which contains the best result so far obtained in this area.

More recently attention has been paid to the problem of establishing Berry-Esseen bounds for L-statistics. We mention the work of Bjerve (1977), Helmers (1977, 1981, 1982), Serfling (1980) and van Zwet (1983). These authors obtained Berry-Esseen bounds for L-statistics for the case of bounded weights. The purpose of this paper is to derive a Berry-Esseen bound for L-statistics with unbounded weight functions.

Let ϕ denote the standard normal df and define F^{-1} by

$$F^{-1}(s) = \inf\{x: F(x) \geq s\} \text{ for } 0 < s < 1.$$

THEOREM 1. Suppose there exists numbers $\delta > 0$, $\epsilon > 0$ and $K > 0$ such that

(I) the function J satisfies a Lipschitz condition of order 1 on $[\epsilon, 1-\epsilon]$, whereas on neighbourhoods $(0, \epsilon)$ and $(1-\epsilon, 1)$ of zero and one, J is twice differentiable with second derivative J'' , satisfying

$$(1.4) \quad |J''(s)| \leq K[s(1-s)]^{-2}$$

(II) the inverse F^{-1} satisfies

$$(1.5) \quad |F^{-1}(s)| \leq K[s(1-s)]^{-\frac{1}{4}+\delta} \text{ for } 0 < s < 1$$

and

$$(1.6) \quad |F^{-1}(s_1) - F^{-1}(s_2)| \leq K|s_1 - s_2| [(s_1(1-s_1))^{-\frac{5}{4}+\delta} + (s_2(1-s_2))^{-\frac{5}{4}+\delta}]$$

for $0 < s_1, s_2 < \epsilon$ and $1-\epsilon < s_1, s_2 < 1$. Then $\sigma^2(J, F) > 0$ where

$$(1.7) \quad \sigma^2(J, F) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(F(x))J(F(y))(F(\min(x,y)) - F(x)F(y)) dx dy$$

implies that

$$(1.8) \quad \sup_x |F_n^*(x) - \phi(x)| = O(n^{-\frac{1}{2}}), \text{ as } n \rightarrow \infty.$$

The theorem allows weight functions J tending to infinity in the neighbourhood of 0 and 1 at a logarithmic rate. An example is provided by the weight function ϕ^{-1} , the normal quantile function. Then T_n is an asymptotically efficient L-estimator of normal scale.

Our method of proof resembles those of van Zwet (1977) and Does (1982) as these authors also combine smoothing techniques with appropriate conditioning arguments.

In section 2 we prove the theorem. The proofs of a number of lemmas are omitted, but these may be found in Helmers & Hušková (1984).

2. PROOF

Let, for any $n \geq 1$, $(U_{1:n}, \dots, U_{n:n})$ denote the order statistics corresponding to a sample of size n from the uniform distribution on $(0, 1)$. For any integer $1 \leq m \leq [en]$, let $V = (V_{1:m-1}, \dots, V_{m-1:m-1})$, $Z = (Z_{1:n-2m}, \dots, Z_{n-2m:n-2m})$ and $W = (W_{1:m-1}, \dots, W_{m-1:m-1})$ be vectors of order statistics corresponding to samples of sizes $m-1$, $n-2m$, and $m-1$ from the uniform distribution on $(0, 1)$ and let V, Z and $W, U_{m:n}$, and $U_{n-m+1:n}$ be independent. Then the joint distribution of $(U_{1:n}, \dots, U_{n:n})$ is the same as that of

$$(2.1) \quad U_{m:n}, V_{1:m-1}, \dots, U_{m:n}, V_{m-1:m-1}, U_{m:n}$$

$$\begin{aligned} & (U_{n-m+1:n} - U_{m:n})Z_{1:n-2m} + U_{m:n}, \dots, (U_{n-m+1:n} - U_{m:n})Z_{n-2m:n-2m} + \\ & + U_{m:n}, U_{n-m+1:n}, (1 - U_{n-m+1:n})W_{1:m-1} + U_{n-m+1:n}, \dots, \\ & (1 - U_{n-m+1:n})W_{m-1:m-1} + U_{n-m+1:n}. \end{aligned}$$

Since the joint distribution of $X_{i:n}$, $i = 1, \dots, n$ is the same as that of $F^{-1}(U_{i:n})$, $i = 1, \dots, n$ it follows directly from (2.1) that the distribution of T_n (cf.(1.1)) can be identified with that of

$$(2.2) \quad T_{1n}(U_{m:n}) + \int_{\frac{m-1}{n}}^{\frac{m}{n}} J(s) ds F^{-1}(U_{m:n}) + T_{2n}(U_{m:n}, U_{n-m+1:n}) \\ + \int_{\frac{n-m}{n}}^{\frac{n-m+1}{n}} J(s) ds F^{-1}(U_{n-m+1:n}) + T_{3n}(U_{n-m+1:n})$$

where

$$(2.3) \quad T_{1n}(U_{m:n}) = \sum_{i=1}^{m-1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s) ds F^{-1}(V_{i:m-1} U_{m:n})$$

$$(2.4) \quad T_{2n}(U_{m:n}, U_{n-m+1:n}) = \sum_{i=1}^{n-2m} \int_{\frac{i+m-1}{n}}^{\frac{i+m}{n}} J(s) ds \cdot F^{-1}(Z_{i:n-2m}$$

$$(U_{n-m+1:n} - U_{m:n}) + U_{m:n})$$

and

$$(2.5) \quad T_{3n}(U_{n-m+1:n}) = \sum_{i=1}^{m-1} \int_{\frac{i+n-m}{n}}^{\frac{i+n-m+1}{n}} J(s) ds F^{-1}(W_{i:m-1}(1 - U_{n-m+1:n}) + U_{n-m+1:n}).$$

Clearly, the r.v.'s $T_{1n}(U_{m:n})$, $T_{2n}(U_{m:n}, U_{n-m+1:n})$ and $T_{3n}(U_{n-m+1:n})$ are conditionally independent, conditionally given $U_{m:n} = u$ and $U_{n-m+1:n} = v$ for any $0 < u < v < 1$. This fact will be crucial in what follows.

Define, for $\frac{m}{n} \leq s \leq \frac{n-m}{n}$, the function ψ_n by

$$(2.6) \quad \psi_n(s) = \int_s^{\frac{n-m}{n}} J(y) dy - \frac{(\frac{n-m}{n} - s)}{\frac{n-2m}{n}} \int_{\frac{m}{n}}^{\frac{n-m}{n}} J(y) dy$$

and note that $\psi_n(\frac{m}{n}) = \psi_n(\frac{n-m}{n}) = 0$. Let Γ_{n-2m} denote the empirical df based on Z_1, \dots, Z_{n-2m} ; i.e. $\Gamma_{n-2m}(s) = (n-2m)^{-1} \sum_{i=1}^{n-2m} I_{(0,s)}(Z_i)$ for $0 < s < 1$, where Z_1, \dots, Z_{n-2m} are independent uniform (0,1) r.v.'s corresponding to the order

statistics $Z_{1:n-2m}, \dots, Z_{n-2m:n-2m}$. Here and elsewhere $I_A(\cdot)$ denotes the indicator of a set A . For any r.v. X , with $0 < \sigma(X) < \infty$, we write \tilde{X} for $X - EX$ and X^* for $(X - EX)/\sigma(X)$.

Similarly as in Helmers (1981;1982) we can write

$$(2.7) \quad T_{2n}(U_{m:n}, U_{n-m+1:n}) = \\ = \int_0^1 \psi_n\left(\frac{m}{n} + \frac{n-2m}{n}s\right) \Gamma_{n-2m}(s) d F^{-1}(U_{m:n} + (U_{n-m+1:n} - U_{m:n})s) + \\ + (n-2m)^{-1} \sum_{i=1}^{n-2m} F^{-1}(U_{m:n} + (U_{n-m+1:n} - U_{m:n})Z_i) \cdot \int_{\frac{m}{n}}^{\frac{n-m}{n}} J(y) dy.$$

To proceed we note that, as J is Lipschitz of order 1 on $[\epsilon, 1-\epsilon]$ (cf. assumption (I)), we can approximate T_{2n} from above and below for sufficiently large n by r.v.'s T_{2n+} and T_{2n-} defined by

$$(2.8) \quad T_{2n+}(U_{m:n}, U_{n-m+1:n}) = \\ = \int_0^1 \left\{ \psi_n\left(\frac{m}{n} + \frac{n-2m}{n}s\right) + \frac{n-2m}{n} (\Gamma_{n-2m}(s) - s) \psi_n'\left(\frac{m}{n} + \frac{n-2m}{n}s\right) \right. \\ \left. + 2^{-1} L \left(\frac{n-2m}{n}\right)^2 (\Gamma_{n-2m}(s) - s)^2 I_{[\epsilon, 1-\epsilon]}(s) \right. \\ \left. + 2^{-1} \left(\frac{n-2m}{n}\right)^2 (\Gamma_{n-2m}(s) - s)^2 \psi_n''\left(\frac{m}{n} + \frac{n-2m}{n}s\right) I_{(0, \epsilon) \cup (1-\epsilon, 1)}(s) \right. \\ \left. + 6^{-1} \left(\frac{n-2m}{n}\right)^3 (\Gamma_{n-2m}(s) - s)^3 \psi_n'''\left(\frac{m}{n} + \frac{n-2m}{n}s\right) (\lambda s + (1-\lambda) \Gamma_{n-2m}(s)) \right. \\ \left. \cdot I_{(0, \frac{\epsilon}{2}) \cup (1-\frac{\epsilon}{2}, 1)}(s) \right\} d F^{-1}(U_{m:n} + (U_{n-m+1:n} - U_{m:n})s) \\ + (n-2m)^{-1} \sum_{i=1}^{n-2m} F^{-1}(U_{m:n} + (U_{n-m+1:n} - U_{m:n})Z_i) \int_{\frac{m}{n}}^{\frac{n-m}{n}} J(y) dy$$

where L is the Lipschitz constant and λ a random point in $[0, 1]$; i.e.

$$(2.9) \quad T_{2n-}(U_{m:n}, U_{n-m+1:n}) \leq T_{2n}(U_{m:n}, U_{n-m+1:n}) \leq T_{2n+}(U_{m:n}, U_{n-m+1:n}).$$

Define (cf.(2.2))

$$(2.10) \quad T_{n+} = T_n + T_{2n+}(U_{m:n}, U_{n-m+1:n}) - T_{2n}(U_{m:n}, U_{n-m+1:n}).$$

In the following lemma we relate T_n^* with T_{n+}^* and T_{n-}^* (cf. Helmers(1981);(1982) for a similar approach).

LEMMA 2.1. *If the assumptions of Theorem 1 are satisfied, then*

$$(2.11) \quad P(T_n^* \leq x) \leq P(T_{n+}^* \leq x_{n+})$$

and

$$(2.12) \quad P(T_n^* \leq x) \geq P(T_{n+}^* \leq x_{n-})$$

for appropriate sequences x_{n+} , $n = 1, 2, \dots$ and x_{n-} , $n = 1, 2, \dots$ satisfying

$$(2.13) \quad x_{n+} = x(1 + O(n^{-\frac{1}{2}})) + O(n^{-\frac{1}{2}})$$

uniformly in x .

PROOF. See Helmers & Hušková (1984). \square

In view of Lemma 2.1 it obviously suffices to show

$$(2.14) \quad \sup_x |P(T_{n+}^* \leq x) - \Phi(x)| = O(n^{-\frac{1}{2}})$$

instead of (1.8). To prove (2.14) we show that for some sufficiently small $\gamma > 0$

$$(2.15) \quad \int_{|t| \leq n^\gamma} |t|^{-1} |\rho_{n+}^*(t) - e^{-\frac{1}{2}t^2}| dt = O(n^{-\frac{1}{2}})$$

and

$$(2.16) \quad \int_{n^\gamma < |t| \leq n^{\frac{1}{2}}} |t|^{-1} |\rho_{n+}^*(t)| dt = O(n^{-\frac{1}{2}}),$$

where ρ_{n+}^* denotes the characteristic function (ch.f) of T_{n+}^* . An application of Esseen's smoothing lemma (see, e.g., Feller (1971), p.538) will then complete the proof of (2.14).

We first prove (2.15). To start with we note that (2.1)-(2.5) and the remark following (2.5) directly yields

$$(2.17) \quad \rho_{n+}^*(t) = E[\phi_{T_{1n}(U_{m:n})}^*(t) \phi_{T_{2n+}(U_{m:n}, U_{n-m+1:n})}^*(t) \phi_{T_{3n}(U_{n-m+1:n})}^*(t) \exp(it\sigma_{n+}^{-1}(E(T_{n+} | U_{m:n}, U_{n-m+1:n}) - ET_{n+})))]$$

where $\sigma_{n+}^2 = \sigma^2(T_{n+})$ and, for any r.v. X with $E|X| < \infty$,

$$(2.18) \quad \phi_X^*(t) = E(\exp(it\sigma_{n+}^{-1}(X-E(X|U_{m:n}, U_{n-m+1:n}))) | U_{m:n}, U_{n-m+1:n}).$$

Note that the expression within square brackets in (2.17) is precisely equal to the conditional ch.f. of T_{n+}^* , where the conditioning is on $U_{m:n}$ and $U_{n-m+1:n}$. The expectation operator E^- in (2.17) refers to the expected value taken w.r.t. $(U_{m:n}, U_{n-m+1:n})$.

We continue with the analysis of $\rho_{n+}^*(t)$. In the next lemma we derive asymptotic approximations for the first and third factor within square brackets in (2.17); i.e. for $\phi_{T_{1n}}^*(u)(t)$ and $\phi_{T_{3n}}^*(v)(t)$ for $0 < u < \epsilon$ and $1-\epsilon < v < 1$

LEMMA 2.2. *If the assumptions of Theorem 1 are satisfied, then for any real t and $0 < u < \epsilon$*

$$(2.19) \quad |\phi_{T_{1n}}^*(u)(t) - 1 + \frac{1}{2}t^2 \sigma_{n+}^{-2} \sigma^2(T_{1n}(u))| = \\ = O(n^{-3/2}(\log n)^3 |t|^3 u^{-3/4+3\delta_m^{3/2}})$$

and

$$(2.20) \quad \sigma^2(T_{1n}(u)) = O(n^{-2}(\log n)^2 u^{-\frac{1}{2}+2\delta_m}).$$

The relations (2.19) and (2.20) remain valid if we replace $T_{1n}(u)$ by $T_{3n}(v)$ and u by $1-v$.

PROOF. See Helmers & Hušková (1984). \square

We also need an asymptotic approximation for $\phi_{T_{2n+}}^*(u,v)(t)$ for $0 < u < \epsilon, 1-\epsilon < v < 1$. Note that r.v. $S_n(u,v)$ appearing in the following lemma corresponds to the leading term in the stochastic expansion (2.8), conditional on $U_{m:n} = u$ and $U_{n-m+1:n} = v$.

LEMMA 2.3. *If the assumptions of Theorem 1 are satisfied, then for any $|t| \leq n^{\frac{1}{2}}$ and $0 < u < \epsilon, 1-\epsilon < v < 1$.*

$$(2.21) \quad |\phi_{T_{2n+}}^*(u,v)(t) - \exp(-\frac{1}{2}t^2 \sigma_{n+}^{-2} \sigma^2(S_n(u,v)))| \\ = O(n^{-\frac{1}{2}}(t^2 + |t|^3) \exp(-\frac{1}{4}t^2 \sigma_{n+}^{-2} \sigma^2(S_n(u,v))) \\ + n^{-1}t^2((F^{-1}(u))^2 + (F^{-1}(v))^2) + n^{-\frac{1}{2}m} |t|(|F^{-1}(u)| + |F^{-1}(v)|))$$

where

$$(2.22) \quad S_n(u,v) = -\left(\frac{n-2m}{n}\right) \int_0^1 J\left(\frac{m}{n} + \frac{n-2m}{n}s\right) (\Gamma_{n-2m}(s) - s) dF^{-1}(u+(v-u)s).$$

PROOF. Taylor expanding the ch.f. of $T_{2n+}(u,v)$ yields for any t and $0 < u < v < 1$

$$(2.23) \quad \phi_{T_{2n+}}^*(u,v)(t) = E[\exp\{it\sigma_{n+}^{-1} S_n(u,v)\} (1+it\sigma_{n+}^{-1} \tilde{Q}_n(u,v))] + \\ + \frac{1}{2} t^2 \sigma_{n+}^{-2} (Q_n(u,v) + |t| \sigma_{n+}^{-1} E|R_n(u,v)|$$

Here S_n is defined in (2.22), whereas Q_n and R_n are the quadratic and third order terms in (2.8). Exploiting the von-Mises statistic structure of $Q_n(u,v)$ and employing a bound for large deviation probabilities for the empirical df, due to Lai (1975), p.827, for the estimation of $E|R_n(u,v)|$ we arrive at (2.21). For details of the proof see Helmers & Hušková (1984). \square

To deal with the fourth factor within square brackets in (2.17) it will be convenient to have

LEMMA 2.4. *If the assumptions of Theorem 1 are satisfied, then*

$$(2.24) \quad E|E(T_{n+}|U_{m:n}, U_{n-m+1:n}) - E T_{n+}|^3 I_{(0,\epsilon)}(U_{m:n}) I_{(1-\epsilon,1)}(U_{n-m+1:n}) \\ = O(n^{-3/2} (\frac{m}{n})^{3/4+3\delta} (\log n)^3).$$

PROOF. See Helmers & Hušková (1984). \square

We are now in a position to complete the proof of (2.15). Take $m = \lfloor n^{1/3} \rfloor$. Application of an exponential bound for uniform order statistics (see, e.g., Lemma A2.1 of Albers, Bickel and van Zwet (1976)) yields

$$(2.25) \quad \int_{|t| \leq n^\gamma} |t|^{-1} |\rho_{n+}^*(t) - E e^{itT_{n+}^*} I_{(0,\epsilon)}(U_{m:n}) I_{(1-\epsilon,1)}(U_{n-m+1:n})| dt \\ = O(n^{-\frac{1}{2}}).$$

Also we obtain with the aid of Theorem 1 of Mason (1981) that

$$(2.26) \quad 0 < \lim_{n \rightarrow \infty} n \sigma_{n+}^2 = \sigma^2(J,F) < \infty.$$

Using (2.17), (2.26) and the Lemma's 2.2, 2.3 and 2.4 we find after some elementary computations for all $|t| \leq n^\gamma$ for some sufficiently small $\gamma > 0$

$$(2.27) \quad |E e^{itT_{n+}^*} I_{(0,\epsilon)}(U_{m:n}) I_{(1-\epsilon,1)}(U_{n-m+1:n}) - e^{-\frac{1}{2} t^2}| \\ \leq |E[(1 - \frac{1}{2} t^2 \sigma_{n+}^{-2} (T_{1n}(U_{m:n}) | U_{m:n})) (1 - \frac{1}{2} t^2 \sigma_{n+}^{-2} \\ \sigma^2(T_{3n}(U_{n-m+1:n}) | U_{n-m+1:n})) \exp(-\frac{1}{2} t^2 \sigma_{n+}^{-2} (S_n(U_{m:n}, U_{n-m+1:n})))] \\ U_{m:n}, U_{n-m+1:n}) (1 + it(E(T_{n+}^* | U_{m:n}, U_{n-m+1:n}) - \frac{1}{2} t^2 \\ (E(T_{n+}^* | U_{m:n}, U_{n-m+1:n}))^2) I_{(0,\epsilon)}(U_{m:n}) I_{(1-\epsilon,1)}(U_{n-m+1:n}))]|$$

$$-e^{-\frac{1}{2}t^2} | + O(n^{-\frac{1}{2}}(t^2 + |t|^3)\exp(-\frac{1}{5}t^2)) + O(n^{-\frac{1}{2}-\frac{1}{2}\delta}).$$

Combining now (2.25) through (2.27) we arrive after some calculations involving conditional moments (cf. Helmers & Hušková (1984)) at (2.15).

Next we prove (2.16). Take $m = [\frac{1}{4}\epsilon n]$. Using (2.17) once more we find for all $|t| \leq n^{\frac{1}{2}}$

$$(2.28) \quad |\rho_{n+}^*(t)| \leq E|\phi_{T_{2n+}}^*(U_{m:n}, U_{n-m+1:n})(t)|.$$

Clearly $T_{2n+}(u, v)$ is the sum of a non-degenerate U-statistic of degree 2 with a kernel, which is bounded by $C(|F^{-1}(u)| + |F^{-1}(v)|)$ for some constant $C > 0$, and a remainder term satisfying $E|R_n(u, v)| = O(n^{-3/2}(|F^{-1}(u)| + |F^{-1}(v)|))$. Hence the argument given in Helmers and van Zwet (1982), p.504-505, cf. their relation (3.10) can essentially be repeated to find that for some sufficiently small $\gamma > 0$

$$(2.29) \quad \int_{n^\gamma < |t| \leq n^{\frac{1}{2}}} |t|^{-1} |\rho_{n+}^*(t)| dt \leq \\ \leq \int_{n^\gamma < |t| \leq n^{\frac{1}{2}}} |t|^{-1} E|\phi_{T_{2n+}}^*(U_{m:n}, U_{n-m+1:n})(t)| dt \\ = O(n^{-\frac{1}{2}} E[|F^{-1}(U_{m:n})|^3 + |F^{-1}(U_{n-m+1:n})|^3 + \\ + (F^{-1}(U_{m:n}))^2 + (F^{-1}(U_{n-m+1:n}))^2 + (|F^{-1}(U_{m:n})| + \\ + |F^{-1}(U_{n-m+1:n})|)]) \\ = O(n^{-\frac{1}{2}})$$

which proves (2.16). This completes the proof of Theorem 1.

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